V Fourier transform

5-1 definition of Fourier Transform

∗ The Fourier transform of a function f(x) is defined as

$$\mathcal{F}\{f(x)\} \Longrightarrow \mathcal{F}(u) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i u x} dx$$

The inverse Fourier transform, $\mathcal{F}^{-1}$, is defined so that

$$f(x) = \mathcal{F}^{-1}\{\mathcal{F}\{f(x)\}\}$$

$$f(x) = \int_{-\infty}^{\infty} \mathcal{F}(u)\, e^{2\pi i u x} du$$

∗ For more than one dimension，the Fourier transform of a function f(x,y,z)

$$\mathcal{F}(u, v, w) = \iiint_{-\infty}^{\infty} f(x, y, z) e^{-2\pi i (ux + vy + wz)}\, dxdydz$$

Note that

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$
$$\vec{u} = u\hat{u} + v\hat{v} + w\hat{w}$$

$ux + vy + wz$ can be considered as a scalar product of $\vec{u}$ and $\vec{r}$, i. e.

$$\vec{u} \cdot \vec{r} = ux + vy + wz$$

, if the unit vectors of $\vec{u}$ and $\vec{r}$ form an orthonormal set.

$$\hat{x} \cdot \hat{u} = 1, \hat{x} \cdot \hat{v} = 0, \hat{x} \cdot \hat{w} = 0$$
$$\hat{y} \cdot \hat{u} = 0, \hat{y} \cdot \hat{v} = 1, \hat{y} \cdot \hat{w} = 0$$
$$\hat{z} \cdot \hat{u} = 0, \hat{z} \cdot \hat{v} = 0, \hat{z} \cdot \hat{w} = 1$$

Therefore,

$$\mathcal{F}\{f(\vec{r})\} = \mathcal{F}(\vec{u}) = \int_{-\infty}^{\infty} f(\vec{r})\, e^{-2\pi i \vec{u} \cdot \vec{r}} d\vec{r}$$

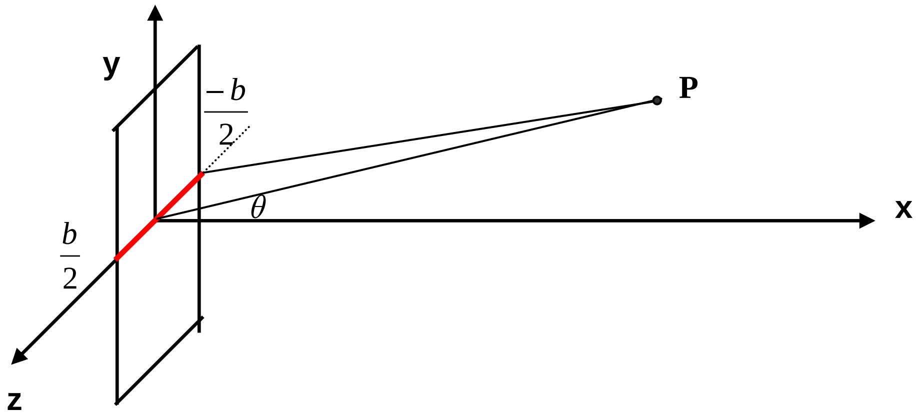
,

and the vector $\vec{u}$ may be considered as a vector in "Fourier transform space"

* The inverse Fourier transform in 3-D space

$$\mathcal{F}^{-1}\{\mathcal{F}(\vec{u})\} = \mathrm{f}(\vec{\mathrm{r}}) = \int_{-\infty}^{\infty} \mathcal{F}(\vec{u})\, \mathrm{e}^{2\pi i \vec{u}\cdot\vec{\mathrm{r}}} \mathrm{d}\vec{u}$$

Consider the diffraction from a single slit



The result from a single slit

$$\tilde{\mathrm{E}} = \frac{\tilde{\xi}'_{\mathrm{L}}}{\mathrm{R}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathrm{e}^{i(\kappa \mathrm{r}-\omega \mathrm{t})}\, \mathrm{dz}$$

Where $\mathrm{r} = \mathrm{R} - \mathrm{z}\sin\theta$

$$\tilde{\mathrm{E}} = \frac{\tilde{\xi}'_{\mathrm{L}}}{\mathrm{R}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \mathrm{e}^{i[(\mathrm{kr}(\mathrm{R}-\mathrm{z}\sin\theta)-\omega \mathrm{t})]} \mathrm{dz}$$

$$\tilde{E} = \frac{\tilde{\xi}'_L}{R} e^{i(kR-\omega t)} \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{-ikz\sin\theta} dz$$

$$\tilde{E} = \frac{\tilde{\xi}'_L}{R} e^{i(kR-\omega t)} \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{-2\pi iz\frac{\sin\theta}{\lambda}} dz$$

The expression is the same as Fourier transform.

$$\mathcal{F}(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi iux} dx$$

$$f(x) = \int_{-\infty}^{\infty} \mathcal{F}(u) e^{2\pi iux} du$$

5-2 Dirac delta function derivation

$$\delta(x-a) = \begin{cases} \infty & \text{for } x = a \\ 0 & \text{for } x \neq a \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x-a) dx = 1$$

derivation：

$$f(x) = \int_{-\infty}^{\infty} \mathcal{F}(u)\, e^{2\pi iux} du$$

$$f(x) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x')\, e^{-2\pi iux'} dx'\right] e^{2\pi iux} du$$

$$f(x) = \int_{-\infty}^{\infty} f(x') \left[\int_{-\infty}^{\infty} e^{-2\pi iu(x'-x)} du\right] dx'$$

Note that

$$f(x) = \int_{-\infty}^{\infty} f(x')\, \delta(x'-x) dx'$$

Therefore,

$$\delta(x' - x) = \int_{-\infty}^{\infty} e^{-2\pi i u (x' - x)} du$$

Set $y = x' - x$,

This leads to

$$\delta(y) = \int_{-\infty}^{\infty} e^{-2\pi i u y} du$$

Similarly,

$$\mathcal{F}(u) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i u x} dx$$

$$\mathcal{F}(u) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \mathcal{F}(u')\, e^{2\pi i u' x} du' \right] e^{-2\pi i u x} dx$$

$$\mathcal{F}(u) = \int_{-\infty}^{\infty} \mathcal{F}(u') \left[\int_{-\infty}^{\infty} e^{2\pi i x (u' - u)} dx \right] du'$$

Note that

$$\mathcal{F}(u) = \int_{-\infty}^{\infty} \mathcal{F}(u')\, \delta(u' - u) du'$$

Therefore,

$$\delta(u' - u) = \int_{-\infty}^{\infty} e^{2\pi i x (u' - u)} dx$$

Set $y = u' - u$,

This leads to

$$\delta(y) = \int_{-\infty}^{\infty} e^{2\pi i x y} dx$$

Comparing with

$$\delta(y) = \int_{-\infty}^{\infty} e^{-2\pi i u y} du$$

This indicates that $\delta(y)$ exhibits a character

$$\delta(y) = \delta(-y)$$

$$\int_{-\infty}^{\infty} e^{2\pi i u y} du = \int_{-\infty}^{\infty} e^{-2\pi i u y} du$$

5-3 A number of general relationships may be written for any function f(x)，real or complex.

Real space	Fourier transform space
f(x)	$F(u)$
f(-x)	$-F(-u)$
f(ax)	$\frac{1}{a}F(\frac{u}{a})$
f(x)+g(x)	$F(u)+G(u)$
F(x-a)	$e^{-2\pi i a u}F(u)$
$\frac{d}{dx}f(x)$	$2\pi i u F(u)$
$\frac{d^n}{dx^n}f(x)$	$(2\pi i u)^n F(u)$

Examples：

(1) $f(ax) \rightarrow \frac{1}{a}\mathcal{F}\left(\frac{u}{a}\right)$

$$\mathcal{F}\{f(ax)\} = \int_{-\infty}^{\infty} f(ax)\, e^{-2\pi i u x} dx$$

Set X=ax

Then

$$\mathcal{F}\{f(ax)\} = \int_{-\infty}^{\infty} f(X)\, e^{-2\pi i u \frac{X}{a}} d\frac{X}{a}$$

$$\mathcal{F}\{f(ax)\} = \frac{1}{a}\int_{-\infty}^{\infty} f(X)\, e^{-2\pi i u \frac{X}{a}} dX$$

$$\mathcal{F}\{f(ax)\} = \frac{1}{a}\,\mathcal{F}\left(\frac{u}{a}\right)$$

(2) $f(x-a) \rightarrow e^{2\pi i a u}\mathcal{F}(u)$

$$\mathcal{F}\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a)\, e^{-2\pi i u x} dx$$

Set X=x-a

Then

$$\mathcal{F}\{f(x-a)\} = \int_{-\infty}^{\infty} f(X)\, e^{-2\pi i u(X+a)} d(X+a)$$

$$\mathcal{F}\{f(x-a)\} = e^{-2\pi i u a} \int_{-\infty}^{\infty} f(X)\, e^{-2\pi i u X} dX$$

$$\mathcal{F}\{f(x-a)\} = e^{-2\pi i u a} \mathcal{F}(u)$$

(3) $\frac{d}{dx} f(x) \rightarrow 2\pi i u \mathcal{F}(u)$

$$\mathcal{F}\left\{\frac{d}{dx} f(x)\right\} = \int_{-\infty}^{\infty} \frac{d}{dx} f(x)\, e^{-2\pi i u x} dx$$

$$\mathcal{F}\left\{\frac{d}{dx} f(x)\right\} = \int_{-\infty}^{\infty} \frac{d}{dx}\left[\int_{-\infty}^{\infty} \mathcal{F}(u')\, e^{2\pi i u' x} du'\right] e^{-2\pi i u x} dx$$

$$\mathcal{F}\left\{\frac{d}{dx} f(x)\right\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \mathcal{F}(u') \left(\frac{d}{dx} e^{2\pi i u' x}\right) du'\right] e^{-2\pi i u x} dx$$

Then

$$\mathcal{F}\left\{\frac{d}{dx} f(x)\right\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \mathcal{F}(u')\, 2\pi i u' e^{2\pi i u' x} du'\right] e^{-2\pi i u x} dx$$

$$\mathcal{F}\left\{\frac{d}{dx} f(x)\right\} = \int_{-\infty}^{\infty} 2\pi i u' \mathcal{F}(u') \left[\int_{-\infty}^{\infty} e^{2\pi i (u'-u) x}\, dx\right] du'$$

$$\mathcal{F}\left\{\frac{d}{dx} f(x)\right\} = \int_{-\infty}^{\infty} 2\pi i u' \mathcal{F}(u') \delta(u'-u)\, du'$$

$$\mathcal{F}\left\{\frac{d}{dx} f(x)\right\} = 2\pi i u \mathcal{F}(u)$$

5-4 Fourier transform and diffraction

(i) point source or point aperture

A small aperture in one dimension can be described as $\delta(x)$ or $\delta(x-a)$. The Fourier transform used to derive Fraunhofer diffraction pattern is illustrated below.

For $\delta(x)$

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x)\, e^{-2\pi i u x} dx$$

$$\mathcal{F}\{\delta(x)\} = e^{-2\pi i u \cdot 0} \int_{-\infty}^{\infty} \delta(x)\, dx$$

$$\mathcal{F}\{\delta(x)\} = e^{-2\pi i u \cdot 0} \cdot 1$$

$$\mathcal{F}\{\delta(x)\} = 1 \cdot 1 = 1$$

The intensity is proportional $|F(u)|^2 = 1$

For $\delta(x-a)$

$$\mathcal{F}\{\delta(x-a)\} = \int_{-\infty}^{\infty} \delta(x-a)\, e^{-2\pi i u x} dx$$

Set X=x-a

Then

$$\mathcal{F}\{\delta(X)\} = \int_{-\infty}^{\infty} \delta(X)\, e^{-2\pi i u (X+a)} d(X+a)$$

$$\mathcal{F}\{\delta(X)\} = e^{-2\pi i u a} \int_{-\infty}^{\infty} \delta(X)\, e^{-2\pi i u X} dX$$

$$\mathcal{F}\{\delta(X)\} = e^{-2\pi i u a} \cdot 1$$

$$\mathcal{F}\{\delta(X)\} = e^{-2\pi i u a}$$

The intensity is proportional $|F(u)|^2 = 1$

Remarks:

The difference between the point source at x=0 and x=a is the phase difference.

(ii) a slit function

$$f(x) = \begin{cases} 0 & \text{when } |x| \geq \frac{b}{2} \\ 1 & \text{when } |x| \leq \frac{b}{2} \end{cases}$$

$$\mathcal{F}\{f(x)\} = \mathcal{F}(u) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i u x} dx$$

$$\mathcal{F}(u) = \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{-2\pi i u x}\, dx$$

$$\mathcal{F}(u) = \left.\frac{e^{-2\pi i u x}}{-2\pi i u}\right|_{-\frac{b}{2}}^{\frac{b}{2}}$$

$$\mathcal{F}(u) = \frac{e^{-2\pi i u \frac{b}{2}} - e^{2\pi i u \frac{b}{2}}}{-2\pi i u}$$

$$\mathcal{F}(u) = \frac{-2i\sin(\pi u b)}{-2\pi i u}$$

$$\mathcal{F}(u) = \frac{\sin(\pi u b)}{\pi u}$$

c.f. the kinematic diffraction from a slit

$$\tilde{E} = \frac{\tilde{\xi}'_L}{R} \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{i(\kappa r - \omega t)}\, dz = \frac{\tilde{\xi}'_L}{R} \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{i(\kappa(R - z\sin\theta) - \omega t)}\, dz$$

if $R \gg z$ (Fraunhofer approximation)

$$\tilde{E} = \frac{\tilde{\xi}'_L}{R} e^{i(\kappa R - \omega t)} \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{-i(\kappa z \sin\theta)}\, dz$$

$$\tilde{E} = \frac{\tilde{\xi}'_L}{R} e^{i(\kappa R - \omega t)} \left.\frac{e^{-i(\kappa z \sin\theta)}}{-i\kappa\sin\theta}\right|_{-\frac{b}{2}}^{\frac{b}{2}}$$

$$\tilde{E} = \frac{\tilde{\xi}_L'}{R} e^{i(\kappa R-\omega t)} \frac{-2i\sin(\frac{\kappa b\sin\theta}{2})}{-i\kappa\sin\theta}$$

$$\tilde{E} = \frac{\tilde{\xi}_L' b}{R} e^{i(\kappa R-\omega t)} \frac{\sin(\frac{\kappa b\sin\theta}{2})}{\frac{\kappa b\sin\theta}{2}}$$

$$\tilde{E} = \frac{\tilde{\xi}_L' b}{R} e^{i(\kappa R-\omega t)} \frac{\sin\beta}{\beta}, \text{ where } \beta = \frac{\kappa b\sin\theta}{2}$$

From the similarity，we obtain $\pi u b$ is equivalent to $\frac{\kappa b\sin\theta}{2}$

$$\pi u b = \frac{\kappa b\sin\theta}{2}$$

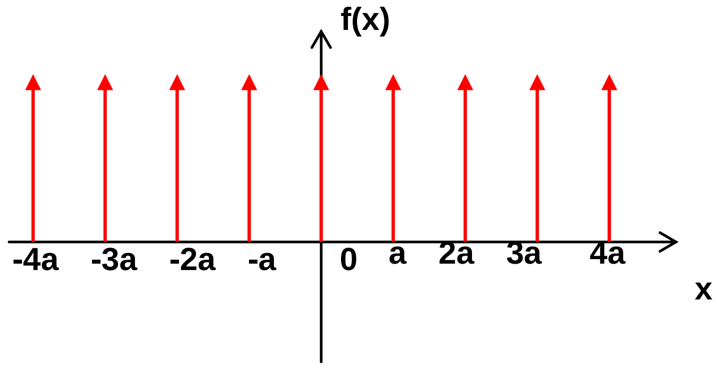
$$\pi u b = \frac{\pi b\sin\theta}{\lambda}$$

Therefore

u is equivalent to $\frac{\sin\theta}{\lambda}$

(iii) a periodic array of narrow slits

$$f(x) = \sum_{n=-\infty}^{\infty} \delta(x - na)$$



The Fourier transform is

$$\mathcal{F}\{f(x)\} = \mathcal{F}(u) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i u x} dx$$

$$\mathcal{F}(u) = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} \delta(x - na) \right] e^{-2\pi i u x} dx$$

$$\mathcal{F}(u) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - na)\, e^{-2\pi i u x} dx$$

$$\mathcal{F}(u) = \sum_{n=-\infty}^{\infty} e^{-2\pi i u n a} \int_{-\infty}^{\infty} \delta(x - na)\, dx$$

$$\mathcal{F}(u) = \sum_{n=-\infty}^{\infty} e^{-2\pi i u n a}$$

Since

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$$

$$\mathcal{F}(u) = \sum_{n=-\infty}^{\infty} e^{-2\pi i u n a}$$

$$\mathcal{F}(u) = \sum_{n=0}^{\infty} (e^{2\pi i u a})^n + \sum_{n=0}^{\infty} (e^{-2\pi i u a})^n - 1$$

$$\mathcal{F}(u) = \sum_{n=0}^{\infty} (e^{2\pi i u a})^n + \sum_{n=0}^{\infty} (e^{-2\pi i u a})^n - 1$$

$$\mathcal{F}(u) = \frac{1}{1 - e^{2\pi i u a}} + \frac{1}{1 - e^{-2\pi i u a}} - 1$$

Discussion

for $e^{-2\pi i u a} \neq 1$

$$\mathcal{F}(u) = \frac{1 - e^{-2\pi i u a} + 1 - e^{2\pi i u a}}{1 - e^{2\pi i u a} + 1 - e^{-2\pi i u a}} - 1$$

$$\mathcal{F}(u) = 1 - 1 = 0$$

for $e^{-2\pi i u a} = 1$

$$\mathcal{F}(u) = \infty$$

It occurs at the condition

$$e^{-2\pi i u a} = \cos(2\pi u a) - i \sin(2\pi u a) = 1$$

$$2\pi u a = 2\pi h$$

,where h is an integer.

$$u a = h$$

In other words,

$$\mathcal{F}(u) = \sum_{h=-\infty}^{\infty} \delta(u a - h)$$

note that

$$\delta(ax) = \frac{\delta(x)}{|a|}$$

Proof:

$$\delta(x) = \begin{cases} \infty & \text{for } x = 0 \\ 0 & \text{for } x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \int_{-\infty}^{\infty} \delta(|a| x)\, dx$$

Set $x' = |a| x$

$$\int_{-\infty}^{\infty} \delta(|a|x)\,dx = \int_{-\infty}^{\infty} \delta(x')\frac{dx'}{|a|} = \frac{1}{|a|}$$

Therefore

$$\delta(ax) = \frac{\delta(x)}{|a|}$$

The Fourier transform of f(x) can be expressed as

$$\mathcal{F}(u) = \sum_{h=-\infty}^{\infty} \delta(ua - h) = \sum_{h=-\infty}^{\infty} \delta\left[a\left(u - \frac{h}{a}\right)\right]$$

$$\mathcal{F}(u) = \frac{1}{a}\sum_{h=-\infty}^{\infty} \delta\left(u - \frac{h}{a}\right)$$

, where $a > 0$

Hence, the Fourier transform is a set of equally spaced delta functions of a period $\frac{1}{a}$

Similarly, a periodic 3-D lattice in real space; (a, b, c)

$$\rho(\vec{r}) = \sum_{m}^{\infty}\sum_{n}^{\infty}\sum_{p}^{\infty} \delta(x - ma, y - nb, z - pc)$$

$$\mathcal{F}\{\rho(\vec{r})\} = \mathcal{F}(u) = \int_{-\infty}^{\infty}\left[\sum_{m}^{\infty}\sum_{n}^{\infty}\sum_{p}^{\infty} \delta(x - ma, y - nb, z - pc))\right] e^{-2\pi i u x} e^{-2\pi i v y} e^{-2\pi i w z} dx$$

$$\mathcal{F}(u) = \frac{1}{abc}\sum_{h}^{\infty}\sum_{k}^{\infty}\sum_{l}^{\infty} \delta\left(u - \frac{h}{a}\right)\delta\left(\nu - \frac{k}{b}\right)\delta\left(\omega - \frac{l}{c}\right)$$

This is equivalent to a periodic lattice in reciprocal lattice $(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$

(iv) Arbitary periodic function

For an arbitrary periodic function

$$\mathrm{f(x)} = \sum_{h=-\infty}^{\infty} \mathcal{F}_h\, e^{2\pi i \frac{h}{a} x}$$

Then

$$\mathcal{F}\{f(x)\} \Rightarrow \mathcal{F}(u) = \int_{-\infty}^{\infty} \left[\sum_{h=-\infty}^{\infty} \mathcal{F}_h\, e^{2\pi i \frac{hx}{a}} \right] e^{-2\pi i u x} dx$$

$$\mathcal{F}(u) = \sum_{h=-\infty}^{\infty} \mathcal{F}_h \int_{-\infty}^{\infty} e^{2\pi i \frac{hx}{a}}\, e^{-2\pi i u x} dx$$

$$\mathcal{F}(u) = \sum_{h=-\infty}^{\infty} \mathcal{F}_h \int_{-\infty}^{\infty} e^{-2\pi i \left(u - \frac{h}{a}\right) x} dx$$

$$\mathcal{F}(u) = \sum_{h=-\infty}^{\infty} \mathcal{F}_h\, \delta\left(u - \frac{h}{a}\right)$$

Hence, the $\mathcal{F}(u)$; i.e. diffracted amplitude, is represented by a set of delta functions equally spaced with separation $\frac{1}{a}$ and each delta function has "weight" $\mathcal{F}_h$ that is equal to the Fourier coefficient.

Supplement # 1

Fourier transform of a Gaussian function is also a Gaussian function.

Suppose that f(x) is a Gaussian function

$$f(x) = e^{-a^2x^2}$$

Then

$$\mathcal{F}[f(x)] = \mathcal{F}(u) = \int_{-\infty}^{\infty} e^{-a^2x^2} e^{-2\pi iux} dx$$

$$= \int_{-\infty}^{\infty} e^{-(a^2x^2+2\pi iux)} dx$$

$$= \int_{-\infty}^{\infty} e^{-\left(ax+\frac{\pi iu}{a}\right)^2} e^{-\left(\frac{\pi u}{a}\right)^2} dx$$

define $\beta = ax + \frac{\pi iu}{a}$

$$\frac{d\beta}{a} = dx$$

$$\mathcal{F}(u) = \int_{-\infty}^{\infty} e^{-\beta^2} e^{-\left(\frac{\pi u}{a}\right)^2} dx$$

$$= \frac{1}{a} e^{-\left(\frac{\pi u}{a}\right)^2} \int_{-\infty}^{\infty} e^{-\beta^2} d\beta$$

$$= \frac{\sqrt{\pi}}{a} e^{-\left(\frac{\pi u}{a}\right)^2}$$

Standard deviation is defined as the range of the variable (x or u) over which the function drops by a factor of $e^{-\frac{1}{2}}$ of its maximum value.

$$f(x) = e^{-a^2x^2}$$

Set $e^{-a^2x^2} = e^{-\frac{1}{2}}$

$$\sigma_x = \frac{1}{\sqrt{2}a}$$

$$\mathcal{F}(u) = \frac{\sqrt{\pi}}{a} e^{-\left(\frac{\pi u}{a}\right)^2}$$

$$\sigma_u = \frac{a}{\sqrt{2}\pi}$$

Hence

$$\sigma_x * \sigma_u = \frac{1}{2\pi}$$

c.f.

$$\Delta x * \Delta p \sim h$$
$$\Delta x * \Delta(\hbar k) \sim h$$
$$\Delta x * \Delta\left(\frac{hk}{2\pi}\right) \sim h$$
$$\Delta x * \Delta k \sim 2\pi$$

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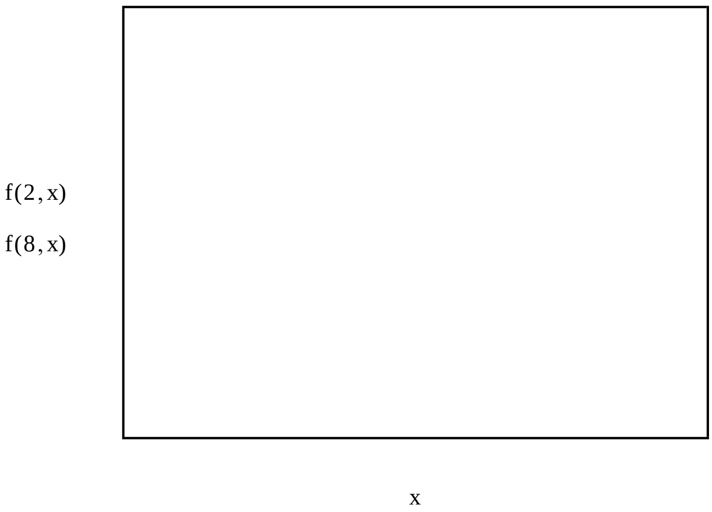
Fourier transform of a Gaussian function

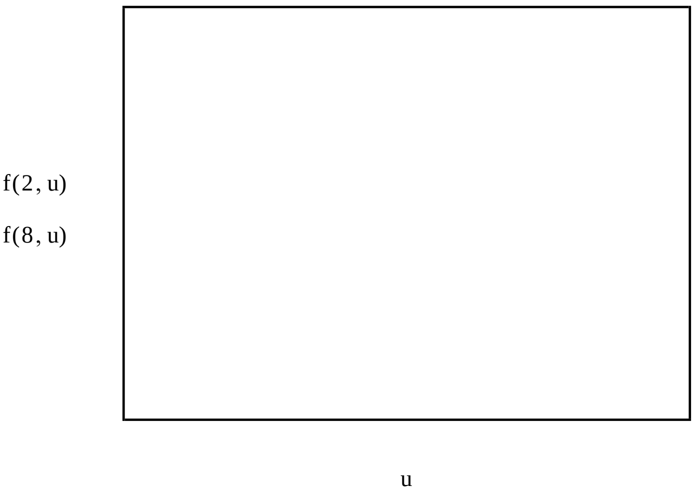
$$x := -2, -1.99 .. 2$$

$$f(a,x) := e^{-a^2 x^2}$$



$$u := -5, -4.99 .. 5$$

$$f(a,u) := \left[\frac{\pi}{(|a|)^2}\right]^{\left(\frac{1}{2}\right)} e^{\left(\frac{-\pi^2}{1}\right)\cdot\left[\frac{u^2}{(|a|)^2}\right]}$$



Supplement #2 Definitions in diffraction

∗ Fourier transform and inverse Fourier transform

System1 :
$$\begin{cases} \mathcal{F}(u) = \int_{-\infty}^{\infty} \mathrm{f(x)}\, \mathrm{e}^{-2\pi i u x} \mathrm{dx} \\ \mathrm{f(x)} = \int_{-\infty}^{\infty} \mathcal{F}(u)\, \mathrm{e}^{2\pi i u x} \mathrm{d}u \end{cases}$$

System2 :
$$\begin{cases} \mathcal{F}(u) = \int_{-\infty}^{\infty} \mathrm{f(x)}\, \mathrm{e}^{-i u x} \mathrm{dx} \\ \mathrm{f(x)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(u)\, \mathrm{e}^{i u x} \mathrm{d}u \end{cases}$$

System3 :
$$\begin{cases} \mathcal{F}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{f(x)}\, \mathrm{e}^{-i u x} \mathrm{dx} \\ \mathrm{f(x)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(u)\, \mathrm{e}^{i u x} \mathrm{d}u \end{cases}$$

System4 :
$$\begin{cases} \mathcal{F}(u) = \int_{-\infty}^{\infty} \mathrm{f(x)}\, \mathrm{e}^{2\pi i u x} \mathrm{dx} \\ \mathrm{f(x)} = \int_{-\infty}^{\infty} \mathcal{F}(u)\, \mathrm{e}^{-2\pi i u x} \mathrm{d}u \end{cases}$$

System5 :
$$\begin{cases} \mathcal{F}(u) = \int_{-\infty}^{\infty} \mathrm{f(x)}\, \mathrm{e}^{i u x} \mathrm{dx} \\ \mathrm{f(x)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(u)\, \mathrm{e}^{-i u x} \mathrm{d}u \end{cases}$$

System6 :
$$\begin{cases} \mathcal{F}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{f(x)}\, \mathrm{e}^{i u x} \mathrm{dx} \\ \mathrm{f(x)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(u)\, \mathrm{e}^{-i u x} \mathrm{d}u \end{cases}$$

∗ relationship among Fourier transform, reciprocal lattice, and diffraction condition.

System	Reciprocal lattice		Diffraction condition
1, 4	$\vec{a}^* = \frac{\vec{b}\times\vec{c}}{\vec{a}\cdot(\vec{b}\times\vec{c})}$ $\vec{b}^* = \frac{\vec{c}\times\vec{a}}{\vec{b}\cdot(\vec{c}\times\vec{a})}$ $\vec{c}^* = \frac{\vec{a}\times\vec{b}}{\vec{c}\cdot(\vec{a}\times\vec{b})}$	$\vec{G}^*_{hkl} = h\vec{a}^* + k\vec{b}^* + l\vec{c}^*$	$\vec{S}' - \vec{S} = \vec{G}^*_{hkl}$ $\vec{\kappa}' - \vec{\kappa} = 2\pi\vec{G}^*_{hkl}$
2, 3 5, 6	$\vec{a}^* = \frac{2\pi\vec{b}\times\vec{c}}{\vec{a}\cdot(\vec{b}\times\vec{c})}$ $\vec{b}^* = \frac{2\pi\vec{c}\times\vec{a}}{\vec{b}\cdot(\vec{c}\times\vec{a})}$ $\vec{c}^* = \frac{2\pi\vec{a}\times\vec{b}}{\vec{c}\cdot(\vec{a}\times\vec{b})}$	$\vec{G}^*_{hkl} = h\vec{a}^* + k\vec{b}^* + l\vec{c}^*$	$2\pi(\vec{S}' - \vec{S}) = \vec{G}^*_{hkl}$ $\vec{\kappa}' - \vec{\kappa} = \vec{G}^*_{hkl}$